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Relativistic theory of magnetoelastic interactions IV. Hereditary processes

Gérard A Maugin
Université de Paris (VI), Départment de Mécanique Théorique (ERA du CNRS), Tour 66, 4 Place Jussieu, 75230 Paris, Cedex 05, France

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Abstract. In the fourth part of this work, the case of materials which exhibit hereditary—or memory—effects is examined. By requiring that the general constitutive equations for a so-called 'simple' material obey different principles of formulation such as those of equipresence, material indifference, and fading memory, and studying the restrictions placed upon the response functionals by the second principle of thermodynamics, a complete set of functional constitutive equations is derived. In particular, it is shown that the heat flux is a *functional* with respect to the temperature gradient. This result allows the formulation of a heat conduction law which provides a possible answer to the paradox of infinite propagation velocity of thermal disturbances in relativity. By examining the limiting case of steady-state processes, the present formulation is shown to incorporate, in a general frame, the results obtained for nonlinear elastic solids in the preceding papers of this series.

1. Introduction

This fourth and last paper of this series, devoted to the special relativistic theory of magnetoelastic interactions, is concerned with so-called hereditary processes. Here *hereditary* is understood in the sense granted by Volterra (1959) in his theory of functionals: the behaviour of the material at a given event point of a particle trajectory in space–time depends explicitly on the previous situations experienced by this particle along its trajectory. One may equally speak of 'memory' effects. This, of course, is in agreement with the deterministic viewpoint adopted here. Furthermore, the explicit dependence on the past *histories* of the different arguments leads to the notion of constitutive *functionals*. This is an approach made quite popular in classical continuum mechanics during the last decade, especially through the works of Coleman *et al* (see references cited hereafter). The notion of functional used is more general than that used by Volterra and is rather similar to that used in the mathematical works of Gâteaux, Fréchet, Hadamard, ... (cf Rall 1971). This paper offers a more mathematical content than the preceding ones because it is impossible to deal seriously with functionals unless one specifies the topological frame in which the continuity and the differentiability of the functionals can be defined. There are only a few works in relativistic continuum physics that follow this type of approach (for instance, Curtis and Lianis 1971, Maugin 1972c, d, 1973f) although the 'functional approach' contains: (i) a possible answer to the paradox presented by the propagation of thermal disturbances at infinite velocity if one assumes a direct relativistic generalization of Fourier's heat conduction law (see Maugin 1974; also, the comments at the end of the third paper of this series: Maugin 1973b); (ii) the representation of a wide class of dissipative effects (such as viscoelasticity). Also, it provides a very general frame for the study of constitutive equations of which simple cases, such as that of nonlinear elastic solids, can be deduced for a certain approximation of the functionals. In a recent paper, McCarthy (1974) has used this approach to generalize our classical theory of micromagnetism (Maugin and Eringen 1972a, b, Maugin 1972b, 1973e). The purpose of this paper is to bring the present relativistic theory of magneto-elastic interactions to a comparable level of generality.

The Clausius–Duhem inequality obtained previously and which represents the local statement of the second principle of thermodynamics is recalled in § 2. A decomposition of the relativistic stress tensor is given which allows, as in the classical theory (see Maugin 1973e), to put in evidence the different contributions to this stress tensor, specifically: (i) the pure 'elastic' stress tensor which would remain even if magnetic effects were discarded; (ii) the effect of the interactions between matter and the magnetization field through the so-called *local* magnetic field; (iii) the effect of the interactions between neighbouring magnetic spins (exchange forces) that also contribute to the stresses. The last two effects would disappear in non-magnetized media. The decomposition so introduced makes easier the following study. The notion of relativistic 'simple' magnetized thermodeformable media and the corresponding general constitutive equations are given in § 3. The important point to be noted is that all constitutive equations are assumed to depend *functionally* on the same set of independent variables (in agreement with the *principle of equipresence*; cf Eringen 1967, chap 5). In particular, the dependent constitutive variables are functionals of the *history* of the temperature gradient. It is this assumption which offers a possible answer to the heat propagation paradox referred to above†. In contrast to the work of McCarthy (1974), the material is here assumed to be an electricity conductor. Restricted forms of the response functionals are given that satisfy the *principle of material frame indifference in relativity* previously given by the author. The response functionals of the material are assumed to obey the *principle of fading memory* whose formulation is due to Coleman (1964; also, Coleman and Noll 1961, Coleman and Dill 1971). This means that the disturbances experienced by the material at distant event points do not have much influence on the behaviour of the material at the present event point. While this hypothesis appears to be physically reasonable, it settles a topological frame for the functionals, which is needed to justify the analytical operations performed on the latter. Each admissible thermodynamical process is assumed to obey the Clausius–Duhem inequality. The consequences of this requirement are explored in § 4. It is thus shown that all constitutive variables are derivable (in a certain functional sense) from the free energy functional which cannot depend explicitly on the present values of the temperature gradient and the electric field. In particular, this is true for the heat flux vector and the conduction current. This is a rather unusual fact which, however, is consistent with the 'functional approach'. Using the notion of *steady-state continuation* introduced by Coleman *et al*, we then show in § 5 that the case of nonlinear elastic media for which constitutive equations were obtained in part II of this work (Maugin 1973a) can be obtained at the limit of the general case studied in § 4. Section 6 is devoted to comments that bring the present results in contact with those of other works.

Finally, it is pointed out that here the hereditary (dissipative) processes are examined within the frame of *phenomenological* physics since we refer to the continuous structure of the matter. One may envisage other types of approach such as one relying on kinetic

† Gurtin and Pipkin (1968) and McCarthy (1970a, b) have recently considered the same hypothesis in classical physics.

theory arguments. In particular, this can be done in order to provide a different solution to the heat propagation paradox as is clearly demonstrated by Stewart (1971 and subsequent works). However, the present 'macroscopic' treatment allows us to examine all dissipative processes (heat flow, viscoelastic behaviour, dissipative magnetic effects) with the same degree of generality in a sufficiently simple manner. A comparable result would require inextricable calculations in a kinetic theory approach.

2. Prerequisites

The notation is that of the three previous parts of this work (Maugin 1972a, 1973a, b, to be referred to as I, II and III respectively)†. The basic *local conservation laws* of the special relativistic theory of magnetoelastic interactions have been given in these papers (also in Maugin 1973c‡) and will not be repeated here. They consist of the continuity equation, the first Cauchy equations and the so-called energy equation obtained by taking the space-like and time-like projections of the conservation of energy–momentum equation, the second Cauchy equations which describe the time evolution of the magnetic spin, and the Maxwell equations in moving deformable matter subject to the Frenkel condition (the electric polarization four vector vanishes). The matter is supposed to be a heat and electricity conductor. Central to the subsequent development is the local expression of the second principle of thermodynamics referred to as the Clausius–Duhem inequality and written as (see equation (III-2.5))

$$\frac{1}{\theta}\left(-\rho(\dot{\psi}+\eta\dot{\theta})-\frac{1}{\theta}\hat{q}^{\beta}\overset{*}{\theta}_{\beta}+\mathscr{E}_{\gamma}j^{\gamma}+t^{(\beta\alpha)}\sigma_{\alpha\beta}+t^{[\beta\alpha]}\nu_{\alpha\beta}+M^{\beta\alpha\gamma}\mathscr{A}_{\alpha\beta\gamma}\right)\geqslant 0, \tag{2.1}$$

in which

$$\sigma_{\alpha\beta}\equiv e_{(\alpha\beta)},\qquad \omega_{\alpha\beta}\equiv e_{[\alpha\beta]},\qquad e_{\alpha\beta}\equiv \mathbf{P}\{u_{\alpha;\beta}\}, \tag{2.2}$$

$$\nu_{\alpha\beta}\equiv\omega_{\alpha\beta}-\Omega_{\alpha\beta}, \tag{2.3}$$

$$\mathscr{A}_{\alpha\beta\gamma}\equiv\mathbf{P}\left\{\Omega_{\alpha\beta;\gamma}+\frac{1}{c^2}\Omega_{\alpha\beta}\dot{u}_{\gamma}+\frac{2}{c^2}u_{[\alpha;|\gamma|}\dot{u}_{\beta]}\right\}, \tag{2.4}$$

$$\psi = e-\eta\theta, \tag{2.5}$$

$$\overset{*}{\theta}_{\beta}\equiv\mathbf{P}\left\{\theta_{,\beta}+\frac{1}{c^2}\theta\dot{u}_{\beta}\right\}. \tag{2.6}$$

The symbols introduced in these equations bear the following physical significance; c: the light velocity in a vacuum; ρ: the proper density of matter; u^{α}: the four velocity; $\dot{u}^{\alpha}$: the four acceleration; $e_{\alpha\beta}$: the relativistic gradient of the velocity; $\sigma_{\alpha\beta}$: the relativistic rate of strain; $\omega_{\alpha\beta}$: the vorticity tensor; $\Omega_{\alpha\beta}$: the precession velocity of the magnetic spin; $\nu_{\alpha\beta}$: the angular velocity of the magnetic spin with respect to the deformable matter; e: the magneto-internal energy per unit of proper mass; θ: the proper thermodynamical temperature; η: the specific entropy; ψ: the magneto-free energy per unit of proper mass; $\overset{*}{\theta}_{\beta}$: the relativistic temperature gradient; $\hat{q}^{\beta}$: the heat flux four vector; $\mathscr{E}_{\gamma}$: the electric field four vector; j^{γ}: the conduction current four vector, $t^{\beta\alpha}$: the relativistic stress tensor;

† Equations of I, II and III are accordingly referred to with a prefix I, II, or III.

‡ Of course, the same basic conservation laws apply whatever the mechanical behaviour (eg, solid, fluid) of the matter is. That is, the theory of magnetoelastic interactions and that of spinning 'ferrofluids' share the same field equations.

$M^{\beta\alpha\gamma}$: the relativistic couple stress tensor ($M^{\beta\alpha\gamma} = -M^{\alpha\beta\gamma}$). $\mathscr{A}_{\alpha\beta\gamma}$ is the complicated kinematical quantity defined by equation (2.4). It generalizes in some way the notion of gradient of the precession velocity $\Omega_{\alpha\beta}$. The latter is such that the proper time evolution of the magnetization four vector (per unit of proper mass) $\tilde{\mathscr{M}}^{\alpha}$ along the worldline ($\mathscr{C}_{X^K}$) of a material particle (X^K) equipped with $\tilde{\mathscr{M}}^{\alpha}$ is given by (see III)†

$$\dot{\tilde{\mathscr{M}}}^{\alpha} = \left(\Omega^{\alpha}_{.\beta} + \frac{1}{c^2}u^{\alpha}\dot{u}_{\beta}\right)\tilde{\mathscr{M}}^{\beta}. \tag{2.7}$$

All four vectors and tensors introduced above are, except for u^{α}, so-called PU tensor fields (cf I). That is, even though they are expressed in full covariant formalism, they are essentially spatial. All Greek indices run from one to four while all Latin indices will take values 1, 2 and 3 only. Semicolons denote covariant derivatives. A superposed dot indicates the invariant derivative (also referred to as the proper time derivative when it applies to tensor fields expressed as functions of the independent variables X^K and s; the latter is the proper time along $\mathscr{C}_{X^K}$) in the direction u^{α}, ie, $\dot{\mathscr{A}} = u^{\alpha}\mathscr{A}_{;\alpha}$. Parentheses and brackets around a set of indices indicate symmetrization and alternation respectively. The short-hand notation $\mathbf{P}\{\ldots\}$ stands for the operation of projection used to obtain PU tensor fields, eg,

$$\mathbf{P}\{u_{\alpha;\beta}\} \equiv P^{.\mu}_{\alpha}u_{\mu;\nu}P^{\nu}_{.\beta}, \qquad P_{\alpha\beta} \equiv g_{\alpha\beta} + \frac{1}{c^2}u_{\alpha}u_{\beta},$$

in which $P_{\alpha\beta}$ is the projection operator and $g_{\alpha\beta}$ is the normal hyperbolic metric of the minkowskian space–time M^4 (see I).

Further, we define the following quantities:

$$\mathfrak{M}^{\alpha}_{.\beta} \equiv \mathbf{P}\{\tilde{\mathscr{M}}^{\alpha}_{.;\beta}\}, \qquad \mathfrak{M}^{\alpha}_{.\beta}u_{\alpha} = 0, \qquad \mathfrak{M}^{\alpha}_{.\beta}u^{\beta} = 0, \tag{2.8}$$

$$M^{\alpha}_{.K} \equiv \mathfrak{M}^{\alpha}_{.\beta}x^{\beta}_{.K}, \qquad M^{\alpha}_{.K}u_{\alpha} = 0, \tag{2.9}$$

as well as the invariants in M^4 (see II)

$$C_{KL} \equiv P_{\alpha\beta}x^{\alpha}_{.K}x^{\beta}_{.L} = C_{LK}, \qquad M_L \equiv P_{\alpha\beta}x^{\alpha}_{.L}\tilde{\mathscr{M}}^{\beta}, \tag{2.10}$$

$$M_{LK} \equiv P_{\alpha\beta}x^{\alpha}_{.L}M^{\beta}_{.K}, \tag{2.11}$$

that, in order to simplify the notation, we shall also refer to as $\boldsymbol{C}$, $\boldsymbol{M}$, and $\mathbf{M}$. We recall that $x^{\alpha}_{.K}$ (also denoted $\boldsymbol{F}$) is the direct gradient of the motion while $X^{K}_{.;\alpha}$ is the inverse gradient of the motion (cf I). The proper time derivatives of the invariants $\boldsymbol{C}$, $\boldsymbol{M}$, and $\mathbf{M}$ have been computed in II (equations (II-3.18)–(II-3.22)). On account of the expressions (2.2)–(2.4), and (2.8) and (2.9), they are:

$$\begin{aligned}
\dot{C}_{KL} &= 2\sigma_{\alpha\beta}x^{\alpha}_{.K}x^{\beta}_{.L},\\
\dot{M}_L &= (\sigma_{\alpha\gamma} + \nu_{\alpha\gamma})\tilde{\mathscr{M}}^{\alpha}x^{\gamma}_{.L},\\
\dot{M}_{LK} &= x^{[\alpha}_{.L}\tilde{\mathscr{M}}^{\beta]}x^{\mu}_{.K}\mathscr{A}_{\alpha\beta\mu} + x^{\alpha}_{.L}M^{\gamma}_{.K}(\sigma_{\alpha\gamma} + \nu_{\gamma\alpha}).
\end{aligned} \tag{2.12}$$

By use of these expressions, an important transformation can be performed on the

† We use here for $\Omega_{\alpha\beta}$ the sign chosen in III and not that used in II.

equation (2.1). Indeed, if one introduces the following decompositions (suggested by the expressions (2.12)) for $t^{\beta\alpha}$ and $M^{\beta\alpha\mu}$,

$$\begin{aligned} t^{\beta\alpha} &= 2\rho \mathring{T}^{KL} x^{\alpha}_{.K} x^{\beta}_{.L} + \rho \mathring{\mathscr{B}}^{L} \tilde{\mathscr{M}}^{\alpha} x^{\beta}_{.L} + \rho \mathring{\mathscr{M}}^{LK} M^{\alpha}_{.K} x^{\beta}_{.L}, \\ M^{\beta\alpha\mu} &= \rho \mathring{\mathscr{M}}^{LK} x^{[\alpha}_{.L} \tilde{\mathscr{M}}^{\beta]} x^{\mu}_{.K}, \\ \mathring{T}^{KL} &= \mathring{T}^{LK}, \end{aligned} \tag{2.13}$$

in which $\mathring{T}^{KL}$ is a symmetric second order tensor, $\mathring{\mathscr{B}}^{L}$ is an axial vector, and $\mathring{\mathscr{M}}^{LK}$ is a general second order tensor, respectively in $\mathbb{E}^3_R$, then, introducing also the following invariants (vectors in $\mathbb{E}^3_R$):

$$\begin{aligned} Q^K &\equiv \rho^{-1} X^{K}_{..\beta} \hat{q}^{\beta}, & \hat{q}^{\beta} &= \rho x^{\beta}_{.K} Q^K, \\ J^K &\equiv \rho^{-1} X^{K}_{..\beta} j^{\beta}, & j^{\beta} &= \rho x^{\beta}_{.K} J^K, \\ \theta_K &\equiv x^{\alpha}_{.K} \mathring{\theta}_{\alpha}, & \mathring{\theta}_{\alpha} &= X^{K}_{..\alpha} \theta_K, \\ \mathscr{E}_K &= x^{\alpha}_{.K} \mathscr{E}_{\alpha}, & \mathscr{E}_{\alpha} &= X^{K}_{..\alpha} \mathscr{E}_K, \end{aligned} \tag{2.14}$$

it is a simple matter of calculation to show that the Clausius–Duhem inequality (2.1) can also be written in the following invariant form:

$$-(\dot{\psi} + \eta\dot{\theta} + \theta^{-1} Q^K \theta_K) + \mathscr{E}_K J^K + \mathring{T}^{KL} \dot{C}_{KL} + \mathring{\mathscr{B}}^{L} \dot{M}_L + \mathring{\mathscr{M}}^{LK} \dot{M}_{LK} \geqslant 0. \tag{2.15}$$

We now define

$$ {}_E t^{\beta\alpha} \equiv 2\rho \mathring{T}^{KL} x^{\alpha}_{.K} x^{\beta}_{.L}, \qquad {}_E t^{\beta\alpha} = {}_E t^{\alpha\beta}, \qquad {}_E t^{\alpha\beta} u_{\alpha} = 0, \tag{2.16}$$

$$ {}_l\mathscr{B}^{\beta} = -\mathring{\mathscr{B}}^{L} x^{\beta}_{.L}, \qquad {}_l\mathscr{B}^{\beta} u_{\beta} = 0, \tag{2.17}$$

$$ \tau^{\alpha\beta} \equiv \rho \mathring{\mathscr{M}}^{LK} x^{\alpha}_{.L} x^{\beta}_{.K}, \qquad \tau^{\alpha\beta} u_{\alpha} = 0, \qquad \tau^{\alpha\beta} u_{\beta} = 0. \tag{2.18}$$

Conversely,

$$\begin{aligned} \mathring{T}^{KL} &= \tfrac{1}{2}\rho^{-1} {}_E t^{\beta\alpha} X^{K}_{..\alpha} X^{L}_{..\beta}, & \mathring{\mathscr{B}}^{L} &= -{}_l\mathscr{B}^{\beta} X^{L}_{..\beta}, \\ \mathring{\mathscr{M}}^{LK} &= \rho^{-1} \tau^{\alpha\beta} X^{L}_{..\alpha} X^{K}_{..\beta}. \end{aligned} \tag{2.19}$$

Thus equations (2.13) can be written as

$$ t^{\beta\alpha} = {}_E t^{\beta\alpha} - \rho_l \mathscr{B}^{\beta} \tilde{\mathscr{M}}^{\alpha} + \tau^{\beta}_{.\mu} \mathfrak{M}^{\alpha\mu}, \tag{2.20}$$

$$ M^{\beta\alpha\mu} = \tilde{\mathscr{M}}^{[\beta} \tau^{\alpha]\mu}. \tag{2.21}$$

It is clear that whenever there is no magnetization field, the equation (2.21) yields zero while equation (2.20) reduces to

$$ t^{\beta\alpha} = t^{\alpha\beta} \equiv {}_E t^{\alpha\beta}. \tag{2.22}$$

For this reason we call ${}_E t^{\beta\alpha}$ the *pure elastic stress tensor*. In the general case, the equation (2.20) gives:

$$ t^{(\beta\alpha)} = {}_E t^{\beta\alpha} - \rho_l \mathscr{B}^{(\beta} \tilde{\mathscr{M}}^{\alpha)} + \tau^{(\beta}_{.\mu} \mathfrak{M}^{\alpha)\mu}, \tag{2.23a}$$

$$ t^{[\beta\alpha]} = -\rho_l \mathscr{B}^{[\beta} \tilde{\mathscr{M}}^{\alpha]} + \tau^{[\beta}_{.\mu} \mathfrak{M}^{\alpha]\mu}. \tag{2.23b}$$

On account of equations (2.23) and (2.2)–(2.3), it can be demonstrated after a somewhat lengthy computation that the inequality (2.1) is also written as

$$\frac{1}{\theta}\left(-\rho(\dot{\psi} + \eta\dot{\theta}) - \frac{1}{\theta}\hat{q}^{\beta}\mathring{\theta}_{\beta} + \mathscr{E}_{\gamma} j^{\gamma} + t^{\beta\alpha} e_{\alpha\beta} - \rho_l \mathscr{B}^{\beta} \dot{\tilde{\mathscr{M}}}_{\beta} + \tau^{\alpha\beta} \dot{\mathfrak{M}}_{\alpha\beta} \right) \geqslant 0. \tag{2.24}$$

(This can be shown by carrying the expressions (2.23) in equation (2.1). Then, for instance, using equation (2.7), we have

$$\rho_l \mathscr{B}^\beta \tilde{\mathscr{M}}^\alpha \Omega_{\alpha\beta} = \rho_l \mathscr{B}^\beta \left(\frac{1}{c^2} \dot{u}_\alpha u_\beta \tilde{\mathscr{M}}^\alpha - \dot{\tilde{\mathscr{M}}}_\beta \right) = -\rho_l \mathscr{B}^\beta \dot{\tilde{\mathscr{M}}}_\beta .$$

Further, computing $\dot{\mathfrak{M}}_{\alpha\beta}$ in a way similar to that followed in computing the variation of $\mathfrak{M}_{\alpha\beta}$ in Maugin (1973c), and using the spin equation (III-2.3), a long calculation that we do not give here leads to

$$-\tau^\beta_{.\mu} \mathfrak{M}^{\alpha\mu} \Omega_{\alpha\beta} + \tilde{\mathscr{M}}^\beta \tau^{\alpha\mu} \mathscr{A}_{\alpha\beta\mu} \equiv \tau^{\alpha\beta} \dot{\mathfrak{M}}_{\alpha\beta}, \qquad \text{QED.)}$$

Comments. The interest for the decompositions (2.13) is threefold. Firstly, they allow us to work with invariants in M^4 instead of tensor-valued functions for the constitutive equations. This proves to be more convenient. Secondly, the decomposition (2.20) which follows from the expressions (2.13) exhibits the different contributions to the relativistic stresses, ie, the pure elastic stresses, the contribution due to the *local* magnetic field (the latter represents the interactions between matter and magnetic spins), and the contribution due to the interactions between neighbouring magnetic spins through $\tau^\beta_{.\mu}$. This will be shown later. Thirdly, it happens that equation (2.20) or (2.23) is formally the relativistic analogue to an equation derived in the classical theory of magnetized deformable solids (Maugin and Eringen 1972b, equation (7.2); Maugin 1973e, equation (2.31)†). The physical interpretation of the elements of the decomposition obtained in the classical case confirms the interpretation mentioned above in the relativistic case. Also, a classical analogue to equation (2.24) has been obtained in the classical theory (Maugin and Eringen 1972a, equation (8.16)). Several further remarks are in order. The equation (2.23*b*) is similar to the classical equation (1.12) of Maugin and Eringen (1972b). It is also a general case of an equation (valid for all mechanical behaviours) which has been derived by the author (Maugin 1973c, equation (4.41)) for the special case of perfect spinning fluids whose constitutive equations were derivable from an internal energy density $e(\rho, \eta, \tilde{\mathscr{M}}^\alpha, \mathfrak{M}_{\alpha\beta})$ when the latter must be Lorentz invariant. Finally, it is to be noted that, whereas the inequality (2.1) contains only *objective* terms (ie, terms that satisfy the principle of material frame indifference in relativity (PMIR), see § 3), the equivalent inequality (2.24) is expressed with some non-objective terms. Obviously, the invariant form (2.15) is entirely expressed by means of objective terms for the PMIR imposes an invariance only on tensor-valued functions in M^4‡. It follows that, if the PMIR is to be considered seriously as a requirement for the 'good' formulation of constitutive equations, and if one intends to determine the response functionals—ie, the functional constitutive equations—which characterize the material and upon which the Clausius–Duhem inequality places restrictions, one must consider either the form (2.1) or the form (2.15) of this inequality. We shall consider the latter for it offers a much simpler formulation.

† It is remarkable that the classical equation corresponding to equation (2.23) has *necessarily and sufficiently* a decomposition of this type as a consequence of the simultaneous use of the principle of virtual work and the principle of objectivity (Maugin 1973e).

‡ In fact, in a previous paper (Maugin 1973f), we have demonstrated that the tensorial quantities $\mathring{\theta}_\beta$, $\mathscr{E}_\gamma$, $\sigma_{\alpha\beta}$, $v_{\alpha\beta}$, and $\mathscr{A}_{\alpha\beta\gamma}$ were objective while the quantities $\omega_{\alpha\beta}$ and $\Omega_{\alpha\beta}$ (considered separately), and the invariant derivatives of PU tensor fields such as $\dot{\tilde{\mathscr{M}}}_\alpha$ and $\dot{\mathfrak{M}}_{\alpha\beta}$ were not objective. However, the proper time derivatives of the invariants are objective; hence $\dot{C}_{KL}$, $\dot{M}_L$, and $\dot{M}_{LK}$ are objective.

3. Simple magnetized thermodeformable media

We recall that the motion of a material particle (X^K) in space–time along its trajectory $(\mathscr{C}_{XK})$ is described by the diffeomorphism

$$x^\alpha = \mathscr{X}^\alpha(X^K, s), \qquad s \in (-\infty, \infty). \tag{3.1}$$

This represents the event point $\boldsymbol{M}(s) \in (\mathscr{C}_{XK})$. The X^K, $K = 1, 2, 3$, refer to the reference state of this particle defined at $\boldsymbol{P}_0(s = \tau_0) \in (\mathscr{C}_{XK})$. In the sequel, we are interested in constructing constitutive equations at event point $\boldsymbol{P}(s = \tau > \tau_0) \in (\mathscr{C}_{XK})$. If the motion (3.1) is known, then all tensor fields introduced in § 2 can be expressed as functions of the independent variables X^K and s at any point $\boldsymbol{M}(s) \in (\mathscr{C}_{XK})$. Then, the ordered array†

$$\{x^\alpha(s), t^{\alpha\beta}(s), M^{\alpha\beta\gamma}(s), j^\gamma(s), \mathscr{E}_\gamma(s), \tilde{\mathscr{M}}^\alpha(s), \psi(s), \eta(s), \theta(s), \hat{q}^\beta(s), h(s)\} \tag{3.2}$$

in which the dependence upon X^K is understood but is not explicitly indicated, will be referred to as a *simple admissible thermodynamic process* if:

(i) it is compatible with the basic conservation laws;

(ii) it processes a finite non-negative temperature θ such that $\theta \in (0, \infty)$ and $\inf \theta = 0$ (we disregard systems which interact with electromagnetic fields and exhibit negative temperature);

(iii) it satisfies the Clausius–Duhem inequality (2.1).

It is clear that if $x^\alpha(s)$ and $\tilde{\mathscr{M}}^\alpha(s)$ are known, then $x^\alpha_{.K}(s)$, $X^K_{.,\alpha}(s)$, $\rho(s)$, $M^\alpha_{.K}(s)$, and $\mathfrak{M}^\alpha_{.\beta}(s)$ are known and that, on account of the equations (2.13) and (2.14), the following ordered array:

$$\{x^\alpha(s), \overset{*}{T}{}^{KL}(s), \overset{*}{\mathscr{B}}{}^K(s), \overset{*}{\mathscr{M}}{}^{LK}(s), J^K(s), \mathscr{E}_K(s), \tilde{\mathscr{M}}^\alpha(s), \psi(s), \eta(s), \theta(s), Q^K(s), h(s)\} \tag{3.3}$$

is a simple admissible thermodynamic process equivalent to (3.2), the inequality (2.1) being replaced by its invariant form (2.15).

However, we do not know the relations that link the *dependent variables* $\overset{*}{T}{}^{KL}(s)$, $\overset{*}{\mathscr{B}}{}^K(s)$, $\overset{*}{\mathscr{M}}{}^{LK}(s)$, $J^K(s)$, $\psi(s)$, $\eta(s)$ and $Q^K(s)$, to those remaining variables of the set (3.3) which are considered as *independent variables*, ie, $x^\alpha(s)$, $\mathscr{E}_K(s)$, $\tilde{\mathscr{M}}^\alpha(s)$ and $\theta(s)$. Establishing these relations is the purpose of constitutive theory. Special classes of constitutive equations have been considered in foregoing parts of this work. Here we shall consider a larger class of constitutive equations, in other words, a larger class of materials, the simpler ones (eg, nonlinear elastic solids) being included as special cases. We have the

Definition. In agreement with the principle of determinism and the axiom of equipresence (cf Eringen 1967, chap 5), we call *relativistic simple magnetized thermodeformable materials* those materials for which the dependent variables, ie $\overset{*}{T}{}^{KL}$, $\overset{*}{\mathscr{B}}{}^K$, $\overset{*}{\mathscr{M}}{}^{LK}$, J^K, ψ, η and Q^K with values at event point $\boldsymbol{P}(s = \tau) \in (\mathscr{C}_{XK})$ are entirely determined by all values taken by the independent variables $x^\alpha(s)$, $\mathscr{E}_\alpha(s)$, $\tilde{\mathscr{M}}^\alpha(s)$ and $\theta(s)$ and, eventually, the *ad hoc* (first order) gradients of the latter, at all past (with respect to $\boldsymbol{P}(s = \tau)$) event points $\boldsymbol{M}(s < \tau)$ along $(\mathscr{C}_{XK})$) and the values taken by the same variables at $\boldsymbol{P}(s = \tau)$ itself.

† h is the specific heat source (cf I and II); it is assumed to be given. Each symbol used in the enumeration (3.2) in fact stands for the set of distinct components of the corresponding tensorial quantity. The same convention holds true in the subsequent development as, for instance, in the list of independent variables present in equation (3.4).

Thus, considering the free energy ψ as a typical dependent variable, we may consider the following type of constitutive equations:

$$\psi(\tau) = \Psi[x^{\alpha}_{.K}(s), \tilde{\mathscr{M}}^{\alpha}(s), \mathfrak{M}^{\alpha}_{.\beta}(s), \theta(s), \mathring{\theta}_{\beta}(s), \mathscr{E}_{\alpha}(s)] \tag{3.4}$$

with $s \in (-\infty, \tau]$. Indeed, $x^{\alpha}_{.K}$ is the first gradient of the motion x^{α}; $\mathfrak{M}^{\alpha}_{.\beta}$ is the first relativistic gradient of $\tilde{\mathscr{M}}^{\alpha}$ (cf equations (2.8)), and $\mathring{\theta}_{\beta}$ is the *ad hoc* quantity that represents the notion of relativistic gradient of the temperature θ (cf equation (2.6)). We do not consider the gradient of $\mathscr{E}_{\alpha}$ although this would not yield any difficulty. An explicit dependence upon X^K is not exhibited but this, too, would not change anything in the following derivation and would not alter the results deduced. $x^{\alpha}(s)$ cannot appear as an independent variable in equation (3.4) for ψ, and all other dependent variables considered here, are Lorentz invariants in M^4. The requirement that ψ be invariant under space–time translations rules out a possible dependence upon x^{α}.

The expression (3.4) represents a *functional*. For instance, we may think of a functional as an integral of functions of the different arguments $x^{\alpha}_{.K}(s)$, $\tilde{\mathscr{M}}^{\alpha}(s)$, etc, over the proper time interval $(-\infty, \tau]$ along $(\mathscr{C}_{X\kappa})$. The result is a function (a scalar-valued function in the case of ψ) with value at $\boldsymbol{P}(s = \tau)$. Note that the arguments are tensor-valued functions in M^4 expressed in a local curvilinear chart x^{α} (cf I); the summation of the values taken by a tensor field at different event points $\boldsymbol{M}(s) \in (\mathscr{C}_{X\kappa})$ involves some difficulties if a different chart is used at each event point $\boldsymbol{M}(s)$. The difficulty is resolved if one imposes that the constitutive equations of the type (3.4) satisfy the so-called *principle of material frame indifference in relativity* (PMIR) as enunciated by the author (Maugin 1972c, d, 1973f). In a non-mathematical form, this principle can be stated as:

PMIR (Maugin 1973f): Constitutive equations of an ideal relativistic continuous deformable medium must be objective; that is, they must be invariant with respect to superposition of an arbitrary local Herglotz–Born rigid body motion.

A *local* Herglotz–Born rigid body motion is such that $\sigma_{\alpha\beta}(s) = 0$ (for all s along $\mathscr{C}_{X\kappa}$) in an open neighbourhood $\mathscr{T}(\mathscr{C}_{X\kappa})$ of $(\mathscr{C}_{X\kappa})$. The mathematical form of the PMIR and its application to different classes of constitutive equations are given in Maugin (1973f). A *necessary* and *sufficient* condition that constitutive equations of the type (3.4) be objective (or satisfy the PMIR) is that they be of the following form‡:

$$\psi(\tau) = \Psi[\boldsymbol{C}(s), \boldsymbol{M}(s), \mathbf{M}(s), \theta(s), \boldsymbol{G}(s), \boldsymbol{E}(s)] \tag{3.5}$$

with $s \in (-\infty, \tau]$. Here $\boldsymbol{C}$, $\boldsymbol{M}$, and $\mathbf{M}$ are the invariants (tensor-valued functions in the reference state defined at $\boldsymbol{P}_0(s = \tau_0) \in \mathscr{C}_{X\kappa}$) defined in equations (2.10) and (2.11). $\boldsymbol{G}$ and $\boldsymbol{E}$ are short hand notations for the set of components of θ_K and $\mathscr{E}_K$ respectively (cf equations (2.14)), ie, $\boldsymbol{G}(s) = \{\theta_K(s)\}$, $\boldsymbol{E}(s) = \{\mathscr{E}_K(s)\}$. Now we see that the process of summation indicated above as an example of functional form can be carried out without difficulty for all arguments appearing in equation (3.5) may be considered as tensor-valued functions in $\mathbb{E}^3_R$ at the *unique* point $\boldsymbol{P}_0(s = \tau_0 < \tau)$ although they still depend on the parameter s which can take all values in the real interval $(-\infty, \tau]$.

† This form replaces that given in II, § 3.2 under the title of *objectivity*. Both forms yield identical results (cf Maugin 1973f).

‡ The proof which is not trivial is given in Maugin (1973f) for $\psi = \Psi[x^{\alpha}_{.K}(s), \theta(s), \mathring{\theta}_{\beta}(s)]$. It is readily extended to the form (3.4) and will not be repeated here.

It is clear that if one introduces the new time-like variable $\xi = \tau - s$, $\xi \in [0, \infty)$, then one can also write

$$\psi(\tau) = \Psi[C^{\tau}(\xi), M^{\tau}(\xi), \mathbf{M}^{\tau}(\xi), \theta^{\tau}(\xi), G^{\tau}(\xi), E^{\tau}(\xi)], \tag{3.6}$$

$\xi \in [0, \infty)$, in which, for instance, the *total* history of C is defined as

$$C^{\tau}(\xi) \equiv C(\tau - \xi) \equiv C(s). \tag{3.7}$$

Similarly general constitutive equations hold for the other dependent variables $\mathring{T}^{KL}$, $\mathring{\mathscr{B}}^{K}$, $\mathring{\mathscr{M}}^{LK}$, J^K, η and Q^K. We say that the functional equations of the type (3.6) represent the constitutive equations of a relativistic *simple homogeneous magnetized thermodeformable medium*. As we shall see below, such a medium in general is dissipative.

4. Consequences of the second principle of thermodynamics

4.1. Functional frame

(i) The next step in the present study consists in giving the expressions of the restrictions placed upon the constitutive functionals Ψ, etc, by the Clausius–Duhem inequality. The answer depends on the nature of the domain and the assumed smoothness of these functionals. We here follow the notions of nonlinear functional analysis given in Rall (1971)†. It is convenient to introduce the following notational device. We define the ordered quadruple Γ and the pair Δ by

$$\Gamma = \{C, M, \mathbf{M}, \theta\}, \qquad \Delta \equiv \{G, E\}. \tag{4.1}$$

Given the different symmetry properties in $\mathbb{E}^3_R$ of the arguments listed in these expressions, Γ and Δ can be regarded as elements in $V_{(19)} = V_{(6)} \oplus V_{(3)} \oplus V_{(9)} \oplus V_{(1)}$, and $V_{(6)} = V_{(3)} \oplus V_{(3)}$ respectively, where $V_{(m)}$ is a real vector space of dimension m. Thus $\Lambda = \{\Gamma(s), \Delta(s); s \text{ fixed}\}$ is an element of $V_{(25)}$. For a given value of the variable s or ξ, the natural norm on $V_{(25)}$ is (Tr = trace, T = transpose)

$$|||\Lambda|||_{V_{(25)}} = (\Lambda \,.\, \Lambda)^{1/2} \equiv [\mathrm{Tr}\, C^2 + M \,.\, M + \mathrm{Tr}(\mathbf{M}\mathbf{M}^{\mathrm{T}}) + \theta^2 + G \,.\, G + E \,.\, E]^{1/2}. \tag{4.2}$$

The symbol $\mathbf{0}$ will denote the zero element in any $V_{(m)}$. The elements Λ which can occur in an admissible process form a set $\mathscr{C}$ that we assume to be a cone in $V_{(25)}$. B denotes the Banach space formed from functions mapping $[0, \infty)$ into $V_{(25)}$. The set of elements of B corresponding to functions with range in $\mathscr{C}$ forms a cone $\mathfrak{C}$ in B. $\mathfrak{C}$ is the domain of definition of the functionals Ψ, etc.

(ii) Let f be an element in B corresponding to a real vector space $V_{(m)}$. Following Coleman and Mizel (1967, 1968a, b) and Coleman and Dill (1971) to whom we refer, we may introduce an *influence* measure μ and consider the Banach function space B such that, if $f \in B$, then $\|f\|_{B(\mu)} < \infty$, where $\| \ldots \|_{B(\mu)}$ is the norm relative to the measure μ‡. $f^{\tau}_{r}(\xi)$ being the restriction of $f^{\tau}(\xi)$—the latter is defined as in equation (3.7)—on $]0, \infty)$, f^{τ}_{r} is in a cone $\mathfrak{C}_r$ in B_r, the latter being the Banach space of functions $f^{\tau}_{r}(\xi)$,

† However, all special notations are defined in the text.

‡ Following Coleman (1964), we could introduce an *influence* function $h(\xi)$ (a positive monotonically decreasing fixed function such that $\xi^2 h(\xi)$ is integrable on $[0, \infty)$), thus giving a structure of Hilbert space to the space of functions f. The approach considered here is more general.

$\xi \in (0, \infty)$ such that $\|f_r\|_r = \|fc_{(0,\infty)}\|$, where $c_{(0,\infty)}$ is the characteristic function on $(0, \infty)$. Then the norm of f in B is

$$\|f\|_B = \|f_r^\tau\|_r + \|\|f(\xi = 0)\|\|,$$

and $B = V_{(m)} \oplus B_r$ in both algebraic and topological senses. Then it is further supposed that:

(i) f is differentiable in the classical sense at $\xi = 0$;

(ii) f_r^τ is an absolutely continuous function on $(0, \infty)$;

(iii) the *constant* history† f^+ on $[0, \infty)$ whose value is

$$f^+(\xi) \equiv f^\tau(0), \qquad \xi \in [0, \infty),$$

is in $\mathfrak{C}$;

(iv) the following time derivative is also in $\mathfrak{C}$:

$$\frac{\mathrm{d}}{\mathrm{d}\tau} f^\tau(\xi) = -\frac{\mathrm{d}}{\mathrm{d}\xi} f^\tau(\xi). \tag{4.3}$$

Furthermore, for any f in $\mathfrak{C} \subset B$, we define the *difference* history f_d^τ and the integrated history $\bar{f}^\tau$ up to time $s = \tau$ by

$$f_\mathrm{d}^\tau(\xi) \equiv f^\tau(\xi) - f^\tau(0) = f(\tau - \xi) - f(\tau), \tag{4.4}$$

and

$$\bar{f}^\tau(\xi) \equiv \int_0^\xi f^\tau(\zeta)\, \mathrm{d}\zeta, \tag{4.5}$$

respectively. Then, if a superposed dot here means the left-hand derivative of $f(s)$ at $s = \tau$—which is the same as the right-hand derivative of $f^\tau(\xi)$ at $\xi = 0$ up to a sign—ie,

$$\dot{f}(\tau) = \lim_{h \to 0_-} \left(\frac{f(\tau) - f(\tau - h)}{h} \right) = -\lim_{\xi \to 0_+} \left(\frac{f^\tau(\xi) - f^\tau(0)}{\xi} \right) = -\frac{\mathrm{d}}{\mathrm{d}\xi} f^\tau(\xi) \bigg|_{\xi = 0}, \tag{4.6}$$

we have

$$\frac{\mathrm{d}}{\mathrm{d}\tau} f_\mathrm{d}^\tau(\xi) = \frac{\mathrm{d}}{\mathrm{d}\tau} f^\tau(\xi) - \dot{f}(\tau), \tag{4.7}$$

and

$$\frac{\mathrm{d}}{\mathrm{d}\tau} \bar{f}^\tau(\xi) = -f_\mathrm{d}^\tau(\xi), \qquad \frac{\mathrm{d}}{\mathrm{d}\xi} \bar{f}^\tau(\xi) = f^\tau(\xi). \tag{4.8}$$

With the hypotheses made above concerning $f \in \mathfrak{C}$, we suppose that a typical functional‡ $\psi = \Psi[f]$ is continuously *Fréchet differentiable* on $\mathfrak{C}$ for each $f \in \mathfrak{C}$ and every $g \in B$ with $f + g \in \mathfrak{C}$. The total Fréchet derivative at f is noted $\delta\Psi[f; g]$. It is a *linear* functional defined and continuous on $\mathfrak{C} \times \mathscr{E}$ ($\mathscr{E}$ is the subspace of B spanned by $\mathfrak{C}$); it is a *linear* functional of g for each f. In particular, according to the chain rule for the 'theory of fading memory' (see Coleman and Dill 1971), the time derivative of ψ defined as in

† $\mathbf{1}^+(\xi)$ and $\mathbf{0}^+(\xi)$ are the *constant unit* and *zero* histories, the values of which are respectively one and zero on $[0, \infty)$.

‡ The notation used for functionals is the following: $\Psi[A; B|A; C]$ means that Ψ is a *general functional* of A on $(0, \infty)$ and a *classical function* of $A(\xi = 0)$; it is a *linear functional* of B on $[0, \infty)$ and a *linear function* of $C(\xi = 0)$. $\Psi[A; B]$ represents a general functional of A on $[0, \infty)$ and a linear functional of B on $[0, \infty)$.

equation (4.6) is related to the total Fréchet derivative of Ψ by the simple relation

$$\dot{\psi} = \delta\Psi[f;\dot{f}]. \tag{4.9}$$

Depending on the form assumed for Ψ, we have the following results useful in the sequel:

Lemma 1. If $\psi(\tau) = \Psi[f_d^\tau(\xi)|f^\tau(0)]$, then

$$\dot{\psi}(\tau) = D_f\Psi[f_d^\tau|f^\tau(0)]\,.\,\dot{f}(\tau) - \delta\Psi\left[f_d^\tau;\frac{df^\tau(\xi)}{d\xi}\bigg|f^\tau(0)\right], \tag{4.10}$$

with

$$D_f\Psi \equiv \partial_f\Psi - \nabla_f\Psi. \tag{4.11}$$

Here $\partial_f\Psi$ is the partial derivative of Ψ considered as a classical function of the independent variable $f^\tau(0) \equiv f(\tau)$. $\nabla_f\Psi$ is the *functional gradient* of Ψ which is defined by

$$\nabla_f\Psi[f_d^\tau|f^\tau(0)] \equiv \delta\Psi[f_d^\tau;\mathbf{1}^+(\xi)|f^\tau(0)]. \tag{4.12}$$

$D_f\Psi$ is called the *instantaneous derivative* of Ψ (with fixed past history).

Lemma 2. If $\psi(\tau) = \Psi[\bar{f}^\tau(\xi)|f^\tau(0)]$, then

$$\dot{\psi}(\tau) = \nabla_{\bar{f}}\Psi[\bar{f}^\tau|f^\tau(0)]\,.\,f(\tau) + \partial_f\Psi[\bar{f}^\tau|f^\tau(0)]\,.\,\dot{f}(\tau) - \delta\Psi\left[\bar{f}^\tau;\frac{d}{d\xi}f^\tau(\xi)\bigg|f^\tau(0)\right], \tag{4.13}$$

wherein

$$\nabla_{\bar{f}}\Psi[\bar{f}^\tau|f^\tau(0)] \equiv \delta\Psi[\bar{f}^\tau;\mathbf{1}^+(\xi)|f^\tau(0)]. \tag{4.14}$$

The proof of lemma 1 is immediate for, on account of equation (4.9), we have

$$\dot{\psi}(\tau) = \delta\Psi\left[f_d^\tau;\frac{df_d^\tau}{d\tau}\bigg|f^\tau(0)\right] + \partial_f\Psi[f_d^\tau|f^\tau(0)]\,.\,\dot{f}(\tau).$$

On account of its linearity with respect to its second argument, the term $\delta\Psi$ is transformed with the aid of equation (4.7) and the definition (4.12).

The proof of lemma 2 goes as follows. Using (4.9), we have

$$\dot{\psi}(\tau) = \delta\Psi\left[\bar{f}^\tau;\frac{d\bar{f}^\tau(\xi)}{d\tau}\bigg|f^\tau(0)\right] + \partial_f\Psi[\bar{f}^\tau|f^\tau(0)]\,.\,\dot{f}(\tau). \tag{4.15}$$

Taking account of the first of equations (4.8), the definition of $f_d^\tau(\xi)$, and the definition (4.14), we successively have

$$\delta\Psi\left[\bar{f}^\tau;\frac{d\bar{f}^\tau(\xi)}{d\tau}\bigg|f^\tau(0)\right]$$
$$= \delta\Psi[\bar{f}^\tau;\,-\dot{f}_d^\tau(\xi)|f^\tau(0)] = \delta\Psi[\bar{f}^\tau;f^\tau(\xi)|f^\tau(0)] + \nabla_{\bar{f}}\Psi[\bar{f}^\tau|f^\tau(0)]\,.\,f(\tau). \tag{4.16}$$

Replacing $f^\tau(\xi)$ by its value given by the second of equations (4.8), and carrying the resulting expression (4.16) into (4.15), we obtain (4.13).

4.2. Constitutive equations

Using the definitions of the preceding section, we can define from the arrays considered in equation (4.1) the functions $\Gamma^\tau(\xi), \xi \in [0, \infty), \Gamma^\tau_d(\xi), \xi \in [0, \infty)$ and $\bar{\Delta}^\tau(\xi), \xi \in [0, \infty)$ with

$$\begin{aligned} &\Gamma^\tau(0) = \Gamma(\tau), \qquad &\Gamma^\tau_d(0) = 0, \\ &\bar{\Delta}^\tau(0) = 0, \qquad &\bar{\Delta}^\tau_d(\xi) \equiv \bar{\Delta}^\tau(\xi). \end{aligned} \tag{4.17}$$

We set

$$\Lambda^\tau(\xi) \equiv \{\Gamma^\tau(\xi), \bar{\Delta}^\tau(\xi)\},$$

and

$$\Lambda^\tau(0) \equiv \{\Gamma^\tau(0), \Delta^\tau(0)\}. \tag{4.18}$$

From equations (4.17), it follows that

$$\Lambda^\tau_d(\xi) \equiv \{\Gamma^\tau_d(\xi), \bar{\Delta}^\tau(\xi)\}, \tag{4.19}$$

and

$$\{\Lambda^\tau_d(\xi), \Lambda^\tau(0)\} = \{\Gamma^\tau_d(\xi), \bar{\Delta}^\tau(\xi), \Gamma^\tau(0), \Delta^\tau(0)\}. \tag{4.20}$$

Obviously, the knowledge of $\Gamma^\tau_d(\xi), \xi \in [0, \infty)$ and $\Gamma^\tau(0)$ is equivalent to that of the total history $\Gamma^\tau(\xi), \xi \in [0, \infty)$. Similarly, the knowledge of $\bar{\Delta}^\tau(\xi), \xi \in [0, \infty)$ and $\Delta^\tau(0)$ is equivalent to that of $\Delta^\tau(\xi), \xi \in [0, \infty)$.

Further, the following ordered array

$$\Sigma = \{\mathring{T}^{KL}, \mathring{\mathscr{B}}^K, \mathring{\mathscr{M}}^{LK}, -\eta, -Q^K/\theta, J^K\} \tag{4.21}$$

represents the set of dependent variables. Clearly Σ has values in $V_{(25)}$ (better, in $V^*_{(25)}$, the dual of $V_{(25)}$). Then, on account of the definitions introduced above, the whole set of constitutive equations (3.5) can be written as

$$\begin{aligned} &\psi(\tau) = \Psi[\Lambda^\tau_d(\xi) | \Lambda^\tau(0)], \\ &\Sigma(\tau) = \hat{\Sigma}[\Lambda^\tau_d(\xi) | \Lambda^\tau(0)], \qquad \xi \in [0, \infty). \end{aligned} \tag{4.22}$$

Also introducing the ordered array

$$\Omega(\tau) = \{\dot{\Gamma}(\tau), \Delta(\tau)\}, \tag{4.23}$$

the inequality (2.15) is written in the following shorthand notation:

$$-\dot{\psi} + \Sigma \, . \, \Omega \geqslant 0. \tag{4.23}$$

In order to compute $\dot{\psi}$, and since ψ is the functional given in equations (4.10), we must specify the topological frame. We suppose that the topological properties assumed in B are those described in § 4.1(ii), but adapted to the particular case corresponding to the original norm (4.2) of $V_{(25)}$. Then Ψ is Fréchet differentiable with respect to its argument $\Lambda^\tau_d(\xi)$ according to the definition (4.9) and the results (4.10) through (4.18). ψ is also assumed to be continuously differentiable (in the classical function sense) with respect to $\Lambda^\tau(0) = \Lambda(\tau)$. It is to be remarked that the continuity and differentiability conditions assumed physically correspond to the material having a so-called *fading memory* (cf Coleman 1964) in the sense that values taken by the arguments of the functionals (4.22) at *distant* past event points $\boldsymbol{M}(s \ll \tau) \in (\mathscr{C}_{X\kappa})$ do not influence much the behaviour of the material at event point $\boldsymbol{P}(s = \tau) \in (\mathscr{C}_{X\kappa})$ or, in other words, the values of the dependent variables ψ and Σ at this point.

Taking account of the definitions (4.1), (4.4) and (4.5), and using the lemmas established above, we can write the proper time derivative of ψ at the event point $\boldsymbol{P}(s = \tau) \in (\mathscr{C}_{X\kappa})$ as

$$\dot{\psi}(\tau) = \Box\Psi \, . \, \Omega + \partial_{\boldsymbol{G}}\Psi \, . \, \dot{\boldsymbol{G}} + \partial_{\boldsymbol{E}}\Psi \, . \, \dot{\boldsymbol{E}} - \delta\Psi\left[\Lambda^\tau_d(\xi); \frac{\mathrm{d}}{\mathrm{d}\xi}\Lambda^\tau(\xi) \middle| \Lambda^\tau(0)\right]. \tag{4.24}$$

Here $\square\Psi$ denotes, in $V_{(25)}$, the array

$$\square\Psi = \{\mathrm{D}_{\boldsymbol{C}}\Psi, \mathrm{D}_{\boldsymbol{M}}\Psi, \mathrm{D}_{\mathbf{M}}\Psi, \mathrm{D}_{\theta}\Psi, \nabla_{\bar{\boldsymbol{G}}}\Psi, \nabla_{\bar{\boldsymbol{E}}}\Psi\}. \tag{4.25}$$

The symbols ∂, D, ∇ and δ have been defined in § 4.1.

The Clausius–Duhem inequality (2.15) or (4.23) being considered at $\boldsymbol{P}(s = \tau) \in (\mathscr{C}_{XK})$, it can be written in the form

$$(\Sigma - \square\Psi) \,.\, \Omega - \partial_{\boldsymbol{G}}\Psi \,.\, \dot{\boldsymbol{G}} - \partial_{\boldsymbol{E}}\Psi \,.\, \dot{\boldsymbol{E}} + \delta\Psi\left[\Lambda_{\mathrm{d}}^{\tau}(\xi); \frac{\mathrm{d}}{\mathrm{d}\xi}\Lambda^{\tau}(\xi)\middle|\Lambda^{\tau}(0)\right] \geqslant 0. \tag{4.26}$$

A straightforward argument which closely parallels that given in Coleman (1964) and McCarthy (1970a) may be used to show that the inequality (4.26) is satisfied for all admissible histories Ω and all admissible rates $\dot{\boldsymbol{G}}$ and $\dot{\boldsymbol{E}}$ if and only if

$$\partial_{\boldsymbol{G}}\Psi = 0, \qquad \partial_{\boldsymbol{E}}\Psi = 0, \tag{4.27}$$

$$\Sigma = \square\Psi. \tag{4.28}$$

The equations (4.27) mean that, in this theory, Ψ cannot depend explicitly on the present values $\boldsymbol{G}(\tau)$ and $\boldsymbol{E}(\tau)$ of the temperature gradient and the electric field so that, on account of the notation introduced above, Ψ reduces to the functional form

$$\psi(\tau) = \Psi[\boldsymbol{C}_{\mathrm{d}}^{\tau}(\xi), \boldsymbol{M}_{\mathrm{d}}^{\tau}(\xi), \mathbf{M}_{\mathrm{d}}^{\tau}(\xi), \theta_{\mathrm{d}}^{\tau}(\xi), \bar{\boldsymbol{G}}^{\tau}(\xi), \bar{\boldsymbol{E}}^{\tau}(\xi) | \boldsymbol{C}(\tau), \boldsymbol{M}(\tau), \mathbf{M}(\tau), \theta(\tau)]. \tag{4.29}$$

On account of the definitions (4.21) and (4.25), the equation (4.28) represents the following set of component equations:

$$\begin{aligned} &\overset{*}{T}{}^{KL} = \mathrm{D}_{C_{KL}}\Psi, \qquad \overset{*}{\mathscr{B}}{}^{K} = \mathrm{D}_{M_K}\Psi, \qquad \overset{*}{\mathscr{M}}{}^{LK} = \mathrm{D}_{\mathrm{M}_{LK}}\Psi, \\ &\eta = -\mathrm{D}_{\theta}\Psi, \\ &Q^K = -\theta\nabla_{\bar{G}_K}\Psi, \qquad J^K = \nabla_{\bar{E}_K}\Psi. \end{aligned} \tag{4.30}$$

On account of the results (4.27) through (4.29), the inequality (4.26) is reduced to

$$\delta\Psi\left[\Lambda_{\mathrm{d}}^{\tau}(\xi); \frac{\mathrm{d}}{\mathrm{d}\xi}\Lambda^{\tau}(\xi)\middle|\Gamma(\tau)\right] \geqslant 0, \qquad \xi \in [0, \infty), \tag{4.31}$$

where Γ is the quadruple defined in equations (4.1). Introducing with an obvious notation the partial Fréchet derivatives† and taking account of equations (4.19), (4.1) and (4.8), we can write this inequality in the developed form:

$$\delta_{\Gamma}\Psi\left[\Lambda_{\mathrm{d}}^{\tau}(\xi); \frac{\mathrm{d}}{\mathrm{d}\xi}\Gamma^{\tau}(\xi)\middle|\Gamma(\tau)\right] + \delta_{\bar{\boldsymbol{G}}}\Psi[\Lambda_{\mathrm{d}}^{\tau}(\xi); \boldsymbol{G}^{\tau}(\xi) | \Gamma(\tau)] + \delta_{\bar{\boldsymbol{E}}}\Psi[\Lambda_{\mathrm{d}}^{\tau}(\xi); \boldsymbol{E}^{\tau}(\xi) | \Gamma(\tau)] \geqslant 0. \tag{4.32}$$

This is the *general dissipation inequality*. The left-hand side defines the *dissipation* per unit of proper mass.

† The total Fréchet derivative is the sum of the partial Fréchet derivatives (cf Rall 1971).

Finally, carrying the results (4.30) into the relations (2.13) and (2.14), we obtain the following constitutive equations at event point $\boldsymbol{P}(s = \tau) \in (\mathscr{C}_{XK})$:

$$t^{\beta\alpha}(\tau) = \rho(\tau)[2(\mathrm{D}_{C_{KL}}\Psi)x^{\alpha}_{.K}(\tau) + (\mathrm{D}_{M_L}\Psi)\tilde{\mathscr{M}}^{\alpha}(\tau) + (\mathrm{D}_{M_{LK}}\Psi)M^{\alpha}_{.K}(\tau)]x^{\beta}_{.L}(\tau), \tag{4.33}$$

$$M^{\beta\alpha\mu}(\tau) = \rho(\tau)(\mathrm{D}_{M_{LK}}\Psi)x^{[\alpha}_{.L}(\tau)\tilde{\mathscr{M}}^{\beta]}(\tau)x^{\mu}_{.K}(\tau), \tag{4.34}$$

$$\eta(\tau) = -\mathrm{D}_{\theta}\Psi, \tag{4.35}$$

$$\hat{q}^{\beta}(\tau) = -\rho(\tau)\theta(\tau)x^{\beta}_{.K}(\tau)(\nabla_{\bar{G}_K}\Psi), \tag{4.36}$$

$$j^{\beta}(\tau) = \rho(\tau)x^{\beta}_{.K}(\tau)(\nabla_{\bar{E}_K}\Psi). \tag{4.37}$$

Then we can gather the results of our investigation in the following statement:

Theorem. If follows from the second principle of thermodynamics and the assumptions made with regard to the continuity and the differentiability of the functionals involved that the constitutive equations representing the behaviour of an ideal relativistic *electric conductor magnetized thermodeformable medium* which obeys the principle of fading memory are given by equations (4.33) through (4.37), respectively for the relativistic stress, the relativistic couple stress, the specific entropy, the heat flux, and the conduction current, if the specific free energy assumes the general functional form (4.29) which must satisfy the inequality (4.32).

It is also useful to remark that, on account of equations (4.30) and (2.17)–(2.18), the constitutive equations of the local magnetic field ${}_l\mathscr{B}^{\alpha}$ and the spin interaction tensor $\tau^{\alpha\beta}$ are given by

$${}_l\mathscr{B}^{\beta}(\tau) = -(\mathrm{D}_{M_L}\Psi)x^{\beta}_{.L}(\tau), \tag{4.38}$$

$$\tau^{\alpha\beta}(\tau) = \rho(\tau)(\mathrm{D}_{M_{LK}}\Psi)x^{\alpha}_{.L}(\tau)x^{\beta}_{.K}(\tau). \tag{4.39}$$

5. Steady-state processes

Before examining the limiting case of steady-state processes, we perform a transformation on the inequality (4.32). Consider the second term of this inequality. Noting that it is a linear functional with respect to its argument $\boldsymbol{G}^{\tau}(\xi)$ and that the latter can be written as (cf equation (4.4))

$$\boldsymbol{G}^{\tau}(\xi) = \boldsymbol{G}^{\tau}_{\mathrm{d}}(\xi) + \boldsymbol{G}^{\tau}(0),$$

we have

$$\delta_{\bar{G}}\Psi[\Lambda^{\tau}_{\mathrm{d}}(\xi); \boldsymbol{G}^{\tau}(\xi)|\Gamma(\tau)] = \delta_{\bar{G}}\Psi[\Lambda^{\tau}_{\mathrm{d}}(\xi); \boldsymbol{G}^{\tau}_{\mathrm{d}}(\xi)|\Gamma(\tau)] + \delta_{\bar{G}}\Psi[\Lambda^{\tau}_{\mathrm{d}}(\xi); \mathbf{1}^{+}(\xi)|\Gamma(\tau)] \cdot \boldsymbol{G}(\tau).$$

On account of the definition of the functional gradient, the last term is none other than

$$\nabla_{\bar{G}}\psi[\Lambda^{\tau}_{\mathrm{d}}(\xi)|\Gamma(\tau)] \cdot \boldsymbol{G}(\tau) = -\frac{1}{\theta}\boldsymbol{Q}(\tau) \cdot \boldsymbol{G}(\tau).$$

The last equality follows from equation (4.30, part five). The third term in the left-hand side of equation (4.32) can be transformed in the same manner. Collecting these results, we can write this inequality in the following form:

$$\delta_\Gamma \Psi\left[\Lambda_d^\tau(\xi); \frac{d}{d\xi}\Gamma^\tau(\xi)\middle|\Gamma(\tau)\right] + \delta_{\boldsymbol{G}}\Psi[\Lambda_d^\tau(\xi); \boldsymbol{G}_d^\tau(\xi)|\Gamma(\tau)] + \delta_{\boldsymbol{E}}\Psi[\Lambda_d^\tau(\xi); \boldsymbol{E}_d^\tau(\xi)|\Gamma(\tau)]$$

$$-\frac{1}{\theta}\boldsymbol{Q}(\tau)\cdot\boldsymbol{G}(\tau)+\boldsymbol{J}(\tau)\cdot\boldsymbol{E}(\tau) \geqslant 0. \tag{5.1}$$

At this point it is important to remark that the first term is a *linear* functional in $(d/d\xi)\Gamma^\tau(\xi)$ while the second and third terms are *linear* functionals in $\boldsymbol{G}_d^\tau(\xi)$ and $\boldsymbol{E}_d^\tau(\xi)$ respectively.

We now give a mathematical definition of a steady-state process. Let f be one of the functions considered in § 4.1. The *static* or *steady-state continuation* of f by an amount δ is the function $E^{(\delta)}f$ for which—see Coleman and Dill (1971)—($\delta \geqslant 0$)

$$E^{(\delta)}f(\xi) = \begin{cases} f(0), & \xi \in [0, \delta], \\ f(\xi-\delta), & \xi \in (\delta, \infty). \end{cases}$$

Elementary properties of $E^{(\delta)}f$ are

$$E^{(\delta)}f(\xi) = f^{\tau+\delta}(\xi),$$

$$\dot{f}^{\tau+\delta}(\xi) = 0, \qquad \xi \in [0, \delta],$$

$$\lim_{\delta\to\infty} E^{(\delta)}f = f^{+}(\xi),$$

where $f^{+}(\xi)$ is the constant history defined in § 4.1. The application of the limiting process indicated in the last equation to the different arguments that appear in the functionals considered above yields

$$\lim_{\delta\to\infty} E^{(\delta)}\Gamma^\tau(\xi) = \Gamma^{+}(\xi) = \Gamma^\tau(0) = \Gamma(\tau), \qquad \xi \in [0, \infty), \tag{5.2}$$

$$\lim_{\delta\to\infty} E^{(\delta)}\boldsymbol{G}^\tau(\xi) = \boldsymbol{G}^{+}(\xi) = \boldsymbol{G}^\tau(0) = \boldsymbol{G}(\tau), \tag{5.3}$$

$$\lim_{\delta\to\infty} E^{(\delta)}\boldsymbol{G}_d^\tau(\xi) = \boldsymbol{G}^\tau(0)-\boldsymbol{G}^\tau(0) = \boldsymbol{0}^{+}(\xi), \qquad \xi \in [0, \infty), \tag{5.4}$$

$$\lim_{\delta\to\infty} E^{(\delta)}\frac{d}{d\xi}\Gamma^\tau(\xi) = \lim_{\delta\to\infty}\frac{d}{d\xi}[E^{(\delta)}\Gamma^\tau(\xi)] = \boldsymbol{0}^{+}(\xi), \qquad \xi \in [0, \infty).$$

We see that the limit of the steady-state continuation leads to the value of the different functions at $\boldsymbol{P}(s = \tau) \in (\mathscr{C}_{XK})$.

We may define the equilibrium response functions $\mathring{\Psi}$, $\mathring{T}^{KL}, \ldots, \mathring{Q}^K$, and $\mathring{J}^K$, by (cf Coleman 1964)

$$\mathring{\Psi}[\Gamma, \Delta] = \Psi[\Gamma^{+}(\xi), \Delta^{+}(\xi)], \qquad \text{etc.} \tag{5.5}$$

Since Ψ and the other constitutive functionals are assumed to be continuous functions on $\mathfrak{C}$, it follows that

$$\lim_{\delta\to\infty} \Psi[E^{(\delta)}\Gamma, E^{(\delta)}\Delta] = \mathring{\Psi}[\Gamma^\tau(0), \Delta^\tau(0)], \qquad \text{etc.} \tag{5.6}$$

Thus, for a steady-state continuation carried along the whole trajectory $(\mathscr{C}_{XK})$ up to the event point $\boldsymbol{P}(s = \tau)$—δ is taken to go to infinity—the specific free energy functional reduces to a classical function of arguments with values at $\boldsymbol{P}(s = \tau)$. Moreover, the result (4.27) being still true, $\mathring{\Psi}$ cannot depend on $\Delta^\tau(0)$. Hence, on account of the first of equations (4.1), the specific free energy for steady-state processes is of the form:

$$\psi(\tau) = \psi^*(\boldsymbol{C}(\tau), \boldsymbol{M}(\tau), \mathbf{M}(\tau), \theta(\tau)). \tag{5.7}$$

Since ψ is no longer a functional, the instantaneous derivatives reduce to partial derivatives in the classical sense (cf equation (4.11)). It follows that the functional constitutive equations (4.33) through (4.35) reduce to

$$t^{\beta\alpha} = \rho\left[2\frac{\partial\psi^*}{\partial C_{KL}}x^{\alpha}_{.K}+\frac{\partial\psi^*}{\partial M_L}\tilde{\mathcal{M}}^{\alpha}+\frac{\partial\psi^*}{\partial M_{LK}}M^{\alpha}_{.K}\right]x^{\beta}_{.L}, \tag{5.8}$$

$$M^{\beta\alpha\mu} = \rho\frac{\partial\psi^*}{\partial M_{LK}}x^{[\alpha}_{.L}\tilde{\mathcal{M}}^{\beta]}x^{\mu}_{.K}, \tag{5.9}$$

$$\eta = -\frac{\partial\psi^*}{\partial\theta}. \tag{5.10}$$

The case of the heat flux and the conduction current is somewhat different. First, it is to be noted that the first term in the left-hand side of equation (5.1) vanishes for steady-state processes because Ψ is no longer a functional of $\Gamma^{\tau}(\xi)$. The second and third terms vanish automatically for they are linear in $\boldsymbol{G}^{\tau}_{\mathrm{d}}(\xi)$ and $\boldsymbol{E}^{\tau}_{\mathrm{d}}(\xi)$ respectively, and the latter quantities vanish according to the equation (5.4). Hence, for steady-state processes, the dissipation inequality reduces to

$$-\frac{1}{\theta}\mathring{\boldsymbol{Q}}(\tau)\,.\,\boldsymbol{G}(\tau)+\mathring{\boldsymbol{J}}(\tau)\,.\,\boldsymbol{E}(\tau) \geqslant 0 \tag{5.11}$$

or, equivalently, in invariant and covariant component forms,

$$\begin{aligned} &-\frac{1}{\theta}\mathring{Q}^{K}\theta_{K}+\mathring{J}^{K}\mathscr{E}_{K} \geqslant 0,\\ &-\frac{1}{\theta}\mathring{q}^{\beta}\overset{*}{\theta}_{\beta}+\mathring{j}^{\beta}\mathscr{E}_{\beta} \geqslant 0. \end{aligned} \tag{5.12}$$

Here $\mathring{\boldsymbol{Q}}$ and $\mathring{\boldsymbol{J}}$ (and $\mathring{q}^{\beta}$ and $\mathring{j}^{\beta}$) are the steady-state values of the heat flux and the conduction current. According to the general formulae (4.30, parts five and six), these can be *formally* defined as

$$\begin{aligned} \mathring{\boldsymbol{Q}} &= -\theta\lim\{\nabla_{\bar{\boldsymbol{G}}}\Psi[\bar{\boldsymbol{G}}^{\tau}(\xi), \bar{\boldsymbol{E}}^{\tau}(\xi)|\Gamma(0)]\},\\ \mathring{\boldsymbol{J}} &= \lim\{\nabla_{\bar{\boldsymbol{E}}}\Psi[\bar{\boldsymbol{G}}^{\tau}(\xi), \bar{\boldsymbol{E}}^{\tau}(\xi)|\Gamma(0)]\}, \end{aligned} \tag{5.13}$$

where the limit symbol stands for the limiting process defined in equation (5.3) but applied to $\boldsymbol{G}^{\tau}(\xi)$ and $\boldsymbol{E}^{\tau}(\xi)$, ie, $\boldsymbol{G}^{\tau}(\xi)\mapsto\boldsymbol{G}^{+}(\xi) = \boldsymbol{G}(\tau)$ and $\boldsymbol{E}^{\tau}(\xi)\mapsto\boldsymbol{E}^{+}(\xi) = \boldsymbol{E}(\tau)$. We remark that the corresponding limits of $\bar{\boldsymbol{G}}^{\tau}(\xi)$ and $\bar{\boldsymbol{E}}^{\tau}(\xi)$ involve $\boldsymbol{G}(\tau)$ and $\boldsymbol{E}(\tau)$. Hence we can write the expressions (5.13) as the following functions (no longer derivable from ψ at the limit; cf equation (5.7)):

$$\mathring{\boldsymbol{Q}}(\tau) = \boldsymbol{Q}(\Gamma(\tau), \boldsymbol{G}(\tau), \boldsymbol{E}(\tau)), \tag{5.14}$$

$$\mathring{\boldsymbol{J}}(\tau) = \boldsymbol{J}(\Gamma(\tau), \boldsymbol{G}(\tau), \boldsymbol{E}(\tau)), \tag{5.15}$$

wherein Γ stands for the quadruple defined in equation (4.1). These equations are compatible with the general form stated in equation (5.5). However, in contrast with ψ^*, hence with $t^{\beta\alpha}$, $M^{\beta\alpha\mu}$ and η, $\mathring{\boldsymbol{Q}}$ and $\mathring{\boldsymbol{J}}$ still depend on $\boldsymbol{G}(\tau)$ and $\boldsymbol{E}(\tau)$.

Employing an argument used by Coleman and Noll (1961), one may easily show that, if the function defined in equation (5.14) is differentiable with respect to $\boldsymbol{G}$ at $\boldsymbol{G} = \boldsymbol{0}$ with $\boldsymbol{E} = \boldsymbol{0}$, the inequality (5.11) implies the following result by continuity:

$$\boldsymbol{Q}(\Gamma(\tau), \boldsymbol{0}, \boldsymbol{0}) = \boldsymbol{0}. \tag{5.16}$$

Similarly, if the expression (5.15) is differentiable with respect to $\boldsymbol{E}$ at $\boldsymbol{E} = 0$ with $\boldsymbol{G} = \boldsymbol{0}$, then the same inequality implies that

$$\boldsymbol{J}(\Gamma(\tau), \boldsymbol{0}, \boldsymbol{0}) = \boldsymbol{0}. \tag{5.17}$$

That is, in agreement with the last equation, if the material does not conduct electricity, then the equation (5.16) holds, ie, heat does not flow in a steady-state characterized by Γ unless the temperature gradient is not zero. More generally, neither heat nor current flows in a steady state in which both the electric field and the temperature gradient are zero. If $\boldsymbol{E}$ and $\boldsymbol{G}$ are different from zero, then the fact that $\mathring{\boldsymbol{Q}}$ and $\mathring{\boldsymbol{J}}$ in general depend on $\boldsymbol{E}$ and $\boldsymbol{G}$ respectively accounts for the Thomson and Peltier effects.

Finally, the steady-state values of the local magnetic field ${}_l\mathscr{B}^\alpha$ and the spin interaction tensor $\tau^{\alpha\beta}$ whose functional forms are given by equations (4.38) and (4.39) obviously are

$${}_l\mathscr{B}^\beta = -\frac{\partial\psi^*}{\partial M_L} x^\beta_{.L}, \tag{5.18}$$

$$\tau^{\alpha\beta} = \rho\frac{\partial\psi^*}{\partial M_{LK}} x^\alpha_{.L} x^\beta_{.K}. \tag{5.19}$$

Also, the pure 'elastic' stress ${}_E t^{\beta\alpha}$ defined in equations (2.16) is

$${}_E t^{\beta\alpha} = 2\rho\frac{\partial\psi^*}{\partial C_{KL}} x^{(\alpha}_{.K} x^{\beta)}_{.L}. \tag{5.20}$$

6. Comments

6.1. Hereditary processes

The functional constitutive equations derived in § 4 allow us to represent hereditary processes for the mechanical stress $t^{\beta\alpha}$ and the magnetic effects proper to the theory of magnetoelastic interactions (ie, the existence of a local magnetic field and that of interactions between neighbouring spins via the couple stress tensor $M^{\beta\alpha\mu}$ or the tensor $\tau^{\alpha\beta}$), but also for the heat flux and the conduction current. The results—equations (4.33) through (4.37)—are quite general since no specific form is given for the free energy functional Ψ. Simple specific forms are not studied here because of the lack of space but also, because this study would duplicate some other works. For instance, a peculiar study of the constitutive functional (4.30, part one) along the lines of Coleman and Noll (1961) would show that a particular case of this equation may serve to describe visco-elastic solids of the Volterra type (compare with equations (III-4.24) that represented a viscoelastic material of the Kelvin–Voigt type). The same procedure applied to the second and third of equations (4.30) would probably yield rich rewards. However, the most interesting point for relativistic physics is provided by the constitutive equation (4.36) for the heat flux. This functional constitutive equation can be written in the form

$$\hat{q}^{\beta}(\tau) = -x^{\beta}_{.K}(\tau)\tilde{Q}^{K}[\Lambda^{\tau}_{d}(\xi)|\Lambda^{\tau}(0)]. \tag{6.1}$$

Here we have introduced $\theta(\tau)$ and $\rho(\tau)$ in the functional $\tilde{Q}^K$ for $\theta(\tau)$ is already present in the array $\Lambda^{\tau}(0)$ and $\rho(\tau)$ can be written as (cf Maugin 1973g)

$$\rho(\tau) = \rho_{\mathrm{R}}(\det|C_{KL}(\tau)|)^{-1/2}$$

where ρ_{R} is a constant which defines the matter density in the reference configuration described at $\boldsymbol{P}_0(s = \tau_0) \in (\mathscr{C}_{\chi\kappa})$. Thus, apart from the constant ρ_{R} which is unimportant, $\rho(\tau)$ is a function of $C_{KL}(\tau)$. The latter is already included in the array $\Lambda^{\tau}(0)$; hence the functional form (6.1). In another paper (Maugin 1974), by using the principle of fading memory, we have derived a special case of the equation (6.1), specifically:

$$\hat{q}^{\beta}(\tau) = -\chi x^{\beta}_{.K}(\tau)G^{KL}\left\{\theta_L(\tau) + \int_0^{\infty} \frac{\partial}{\partial\xi}\left[\exp\left(-\frac{\xi}{k}\right)\right]\theta_L(\tau-\xi)\,\mathrm{d}\xi\right\} \tag{6.2}$$

in which G^{KL} is the reciprocal metric in $\mathbb{E}^3_{\mathrm{R}}$ and k and χ are positive scalars. The constitutive equation (6.2) which clearly represents a hereditary process holds in *isotropic* media (whereas no symmetry condition is applied in the present paper). In establishing this equation, we have neglected the interactions between the different transport phenomena (for instance, the interactions between viscosity and heat flow). The remarkable fact is that the integral representation (6.2) is the solution of the following differential equation:

$$\hat{q}_{\beta} + k\underset{u}{\pounds}\hat{q}_{\beta} = -\chi\mathring{\theta}_{\beta} \tag{6.3}$$

where $\pounds_u$ indicates the Lie derivative with respect to the four velocity field u^{α}. This is nothing but the constitutive equation† for heat flux postulated by Kranys (1966a, b) with the difference that $\pounds_u\hat{q}_{\beta}$ replaces the term $\mathring{\hat{q}}_{\beta}$ used by Kranys. It follows that the equation (6.3) is *objective* (ie, it satisfies the PMIR) whereas Kranys' equation is not objective (see the proof in Maugin 1973f). Recent studies of the relativistic Cauchy problem (see, for instance, Mahjoub 1971) have shown that expressions of the type (6.3) can resolve the paradox of the propagation of thermal disturbances at infinite velocity. The general case (6.1) thus contains a possible solution to this paradox and the point raised at the end of the previous article of this series (Maugin 1973b) is answered. It is to be pointed out that Gurtin and Pipkin (1968) have provided the same answer to this paradox in classical physics by studying the heat flux in rigid solids with a constitutive equation similar to equation (4.19, part five).

6.2. Steady-state processes

The constitutive equations (5.8) through (5.10) are none other than the constitutve equations for nonlinear elastic solids obtained in II (compare with equations (II-3.40)–(II-3.42). Also, the equations (5.18) and (5.19) are the constitutive equations (II-5.6) and

† Note that, from our viewpoint, the *constitutive* equation is the integral expression (6.2) and not the equation (6.3).

(II-5.2, part two)†. When the matter does not interact with any electromagnetic fields (non-magnetized media), then the only relevant constitutive equation is that of the pure 'elastic' stress ${}_{\mathrm{E}}t^{\beta\alpha}$ given by equation (5.20). This equation was given before by Grot and Eringen (1966). It follows that the general case of simple thermodeformable materials obeying the principle of fading memory contains as a limiting case (obtained by the process of steady-state continuation described above), that of nonlinear magnetized elastic materials. Further, the heat flux vector corresponding to steady-state processes is only a *function* of the temperature gradient (cf equation (5.14)) and not a *functional*. A linear approximation of $\mathring{\boldsymbol{Q}}$ with respect to $\boldsymbol{G}(\tau)$ in equation (5.14) would provide a heat flux constitutive equation with a form similar to that obtained in III (equation (III-4.26)) by using different arguments. Such a form would not resolve the paradox referred to above. Thus we are led to a remark quite similar to that made by Kranys (1966b): a heat flux hereditary process‡ is necessary to solve the paradox; according to the general presentation given in this paper, in this case, all phenomena (stresses, local magnetic field, conduction current, ...) present simultaneously a hereditary process. This may be worded in another fashion: the hereditary processes considered in this paper obviously lead to transport phenomena; all transport phenomena arise simultaneously and in general interact as the functional dependence of each independent variable on the whole set of dependent variables shows.

References

Coleman B D 1964 *Archs. ration. Mech. Analysis* **17** 1–44, 230–54
Coleman B D and Dill E D 1971 *Archs. ration Mech. Analysis* **41** 132–62
Coleman B D and Mizel V J 1967 *Archs. ration Mech. Analysis* **27** 255–74
—— 1968a *Archs. ration Mech. Analysis* **29** 18–31
—— 1968b *Archs. ration Mech. Analysis* **30** 173–96
Coleman B D and Noll W 1961 *Rev. mod. Phys.* **33** 239–49
Curtis H D and Lianis G 1971 *Int. J. Engng. Sci.* **9** 451–68
Eringen A C 1967 *Mechanics of Continua* (New York: Wiley)
Grot R A and Eringen A C 1966 *Int. J. Engng. Sci.* **4** 610–70
Gurtin M E and Pipkin A C 1968 *Archs. ration. Mech. Analysis* **31** 113–26
Kranys M 1966a *Nuovo Cim.* **B 42** 51–62
—— 1966b *Phys. Lett.* **22** 285–6
Mahjoub B 1971 *Ann. Inst. Henri Poincaré* **15** 153–68
Maugin G A 1972a *J. Phys.* A: *Gen. Phys.* **5** 786–802
—— 1972b *J. Phys.* A: *Gen. Phys.* **5** 1550–62
—— 1972c *C. R. Acad. Sci., Paris* A **275** 319–22
—— 1972d *C. R. Acad. Sci., Paris* A **275** 349–52
—— 1973a *J. Phys.* A: *Math., Nucl. Gen.* **6** 306–21
—— 1973b *J. Phys.* A: *Math., Nucl. Gen.* **6** 1647–66
—— 1973c *Ann. Inst. Henri Poincaré* **20** 41–68
—— 1973d *C.R. Acad. Sci., Paris* A **278** 185–8
—— 1973e *J. Mécanique* **13** No 1 1–22
—— 1973f *Ann. Inst. Henri Poicaré* **20** in the press
—— 1973g *C. R. Acad. Sci., Paris* A **278** in the press
—— 1974 *J. Phys.* A: *Math., Nucl. Gen.* **7** 465–84

† However, equations (5.9) and (3.42) differ by a sign because the angular velocities $\Omega_{\alpha\beta}$ used here and in II differ by a sign. The same remarks hold for ${}_{t}\mathscr{B}^{\alpha}$. The quantity $\tau^{\alpha\beta}$ is noted $\mathfrak{M}^{\alpha\beta}$ in II (it is not to be mistaken for the tensor $\mathfrak{M}^{\alpha\beta}$ used in III and in the present article).

‡ Kranys (1966b) refers to *relaxation* processes for he considers differential constitutive equations. For instance, the constant k in equation (6.3) is the *relaxation* constant.

Maugin G A and Eringen A C 1972a *J. math. Phys.* **13** 143–55
—— 1972b *J. math. Phys.* **13** 1334–47
McCarthy M F 1970a *Int. J. Engng. Sci.* **8** 467–74
—— 1970b *Proc. Vibr. Probl.* **2** 123–33
—— 1974 *Int. J. Engng. Sci.* **12** 45–60
Rall L B (ed) 1971 *Nonlinear Functional Analysis and Applications* (New York: Academic Press)
Stewart J M 1971 *Non-equilibrium Relativistic Kinetic Theory* (Berlin: Springer-Verlag)
Volterra V 1959 *Theory of Functionals and of Integral and Integro-differential Equations* (New York: Dover)